Finite-Time Cooperative Engagement

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***Abstract*—A smooth finite-time distributed control architecture is introduced and analyzed for the cooperative engagement problem. Using a time transformation method as well as Lyapunov stability theory, it is shown that the proposed architecture guarantees finite-time cooperative engagement in that the difference between the positions of each agent and a time-varying target, where this difference represents a dynamic equilibrium point, vanishes in *a-priori* given, user-defined finite time. In addition, this finite-time convergence is achieved without dependence on the initial conditions of agents and in the presence of unknown but bounded velocity of the target. Specifically, we first time transformed the proposed smooth finite-time distributed control architecture into an infinite-time (that is, stretched) interval. This time transformation method is then allowed to utilize tools from standard Lyapunov stability theory in which we analyze convergence properties of this architecture and boundedness of local control signals of each agent in this infinite-time interval. While this note focuses on a particular problem in the context of multiagent systems, the proposed time transformation method and the analysis procedure can be used for many other problems, where *a-priori* given, user-defined finite-time convergence is necessary with smooth control laws.**

***Index Terms*—Cooperative engagement, distributed control, finite-time control, networked multiagent systems.**

I. Introduction

A. Literature Review and Motivation

We are rapidly moving toward a future in which vehicle teams (henceforth, referred as networked multiagent systems) will autonomously perform a broad spectrum of operations in both civilian and military environments. These operations include collaborative exploration; environment surveillance; target tracking; search and rescue; nuclear, biological, and chemical attack detection; and cooperative engagement; to name but a few examples. As a consequence, distributed control, which enable networked multiagent systems to work in coherence through local information exchange between agents, has been the focus of thriving research activity during the last two decades (e.g., see [1]–[3] for a throughout coverage of the recent progress in distributed control theory and algorithms).

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Most of the existing distributed control approaches (see, for example, [4]–[9] and references therein) for networked multiagent systems consider asymptotic (respectively, semistable) convergence of the agent states to static equilibrium points of interest (respectively, continuum of equilibria). However, if the equilibrium point under consideration is dynamic such as in applications of time-varying target tracking and engagement, containment control with time-varying leaders, and dynamic data fusion, just to mention a few, it is well-known that asymptotic convergence guarantees degrade to uniform ultimate boundedness around this equilibrium point (see, for example, [10]–[14] and references therein), unless possibly nonsmooth (that is, non-Lipschitzian) distributed control algorithms (discussed later in detail) are utilized. In addition, it should be noted from a practical standpoint that there are many multiagent systems applications like the ones mentioned earlier involving dynamic equilibrium points of interest as well as rendezvous and the ones requiring sequential execution of complex network operations (that is, multiagent automation), where finite-time convergence is desired.

To this end, there are several notable results in the literature that propose distributed control algorithms for networked multiagent systems with finite-time convergence guarantees (see, for example, [15]–[26] and references therein), which utilize and generalize nonsmooth control tools and methods used, for example, in the seminal papers [27], [28]. In addition to finite-time convergence, another advantage of these algorithms is that convergence can be guaranteed with respect to not only static equilibrium points of interest but also dynamic ones. Yet, a drawback of these architectures is that their finite-time guarantees depend on initial conditions of agents. Thus, it is not possible to assign *a-priori*, user-defined finite-time necessary for many practical networked multiagent systems applications while using these algorithms.

For example, military simultaneous strike requires munitions launched from distant locations to arrive at a desired target at a user-defined finite time. As another example, autonomous vehicles are desired to reconfigure their formation within a user-defined finite time when switching tasks between surveillance and target tracking. However, existing results in, for example, [15]–[26] do not give user the flexibility to assign a desired convergence time. Moreover, since these methods achieve finite-time convergence via utilizing nonsmooth local control signals (typically signum functions-based distributed controllers), they may have chattering in their control signals. From this standpoint, while a practice is to smoothen their dynamics, e.g., by replacing signum functions with tangent hyperbolic functions [29], [30], this process leads to the loss of desired finite-time convergence achieved by these algorithms.

B. Contribution

The contribution of this note is the introduction and the analysis of a smooth finite-time distributed control architecture for cooperative engagement problem. To this end, we consider a networked multiagent system consisting of agents locally exchanging information with each other, where a subset of agents can sense the position of a time-varying target. Using a time transformation method as well as the Lyapunov stability theory, it is shown that the proposed architecture guarantees finite-time cooperative engagement in that the difference between the positions of each agent and a time-varying target, where this difference represents a dynamic equilibrium point, vanishes in *a-priori* given, user-defined finite time. In addition, this finite-time convergence is achieved without dependence on the initial conditions of agents and in the presence of unknown but bounded velocity of this target. Specifically, we first time transform the proposed smooth finite-time distributed control architecture having a dynamic equilibrium point into an infinite-time (that is, stretched) interval. This time transformation method is then allowed to utilize tools from the standard Lyapunov stability theory in which we analyze convergence properties of this architecture and boundedness of local control signals of each agent in this infinite-time interval.[1]

Unlike the results reported in, for example, [15]–[26], the proposed architecture guarantees smooth and *a-priori* given, user-defined finite-time convergence. It should be also noted that similar smooth control algorithms are used in [32], [33] for the design of guidance algorithms for sole vehicles using the sliding mode control theory, in [34]–[37] for consensus and consensus-like algorithms for networked multiagent systems with static continuum of equilibria, and in [37]–[39] for networked multiagent systems containment problems with fixed leader dynamics (that is, leaders having predefined dynamics that do not accept external commands).[2] We emphasize here that this note's results go beyond [32]–[39] not only owing to the fact that a networked multiagent system problem having a dynamic equilibrium point is considered (that is, engaging a time-varying target without the need for its dynamics), but also the proposed analysis procedure utilizes a time transformation method along with the Lyapunov stability theory, which differs from the analysis steps in [32]–[39].

In particular, unlike the analysis steps adopted by the authors of [32]–[39], our system-theoretical approach predicated on the time transformation method directly links a broad spectrum of well-established, valuable results in the control systems literature proposed over infinite-time intervals to the analysis over finite-time intervals. This is because of the fact that the resulting time transformed dynamics is on an infinite-time interval, and its solution is identical to the solution of the original dynamical system over the given finite-time interval (see the proofs of Theorems 1 and 2 below). Hence, aforementioned convergence and boundedness properties over the infinite-time interval through, for example, the standard Lyapunov stability theory give us the equivalent properties for the proposed smooth finite-time distributed control architecture. Here, we also would like to note that this time transformation method and our analysis procedure can be used for many other problems beyond the multiagent systems problem considered in this paper via directly making many well-established infinite-time stability results available to researchers working on finite-time control algorithms. Finally, with regard to other relevant works utilizing nonsmooth distributed control algorithms (see, for example, [23]–[25] and references therein), we also note that the proposed approach does not require knowledge of the upper bound on the velocity of the unknown target and, as discussed earlier, our finite-time convergence can be assigned *a-priori* without dependence on the initial conditions of agents.

[1]The results of [31] may be viewed as authors' earlier work. Yet, only consensus problem with static continuum of equilibria is studied there, unlike this paper focusing on engagement problem with a dynamic equilibrium point.

[2]There are also other studies by the authors of [40], [41], which extend some of these cited results. However, these studies do not derive any conditions to show that the resulting control signals are bounded.

II. Mathematical Preliminaries

In this paper, $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively, denote the set of real numbers, the set of $n\times 1$ real column vectors, and the set of $n\times m$ real matrices; $\mathbb{R}_+$ denotes the set of positive real numbers; $\mathbb{R}_+^{n\times n}$ (respectively, $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n\times n$ positive-definite (respectively, nonnegative-definite) real matrices; $\mathbb{Z}$ denotes the set of integers; $\mathbb{Z}_+$ (respectively, $\overline{\mathbb{Z}}_+$) denotes the set of positive (respectively, nonnegative) integers; and $\mathbf{1}_n$ and I_n, respectively, denote the $n\times 1$ vector of all ones and the $n\times n$ identity matrix. We also write $(\cdot)^{\mathrm{T}}$ for transpose, $(\cdot)^{-1}$ for inverse, $\|\cdot\|_2$ for the Euclidian norm, $\lambda_{\min}(A)$ (respectively, $\lambda_{\max}(A)$) for the minimum (respectively, maximum) eigenvalue of the Hermitian matrix A, $\lambda_i(A)$ for the ith eigenvalue of A (A is symmetric and the eigenvalues are ordered from least to greatest value), $\mathrm{diag}(a)$ for the diagonal matrix with the vector a on its diagonal, and $[A]_{ij}$ for the entry of the matrix A on the ith row and jth column.

We now recall some notions from graph theory (see [3], [42] for details). An undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_\mathcal{G}=\{1,\ldots,n\}$ of nodes and a set $\mathcal{E}_\mathcal{G}\subset\mathcal{V}_\mathcal{G}\times\mathcal{V}_\mathcal{G}$ of edges. If $(i,j)\in\mathcal{E}_\mathcal{G}$, then the nodes i and j are neighbors and the neighboring relation is indicated with $i\sim j$. The degree of a node is given by the number of its neighbors. Letting $d_i\in\mathbb{Z}_+$ be the degree of node i, the degree matrix of a graph $\mathcal{G}$, $\mathcal{D}(\mathcal{G})\in\mathbb{R}_+^{n\times n}$, is given by $\mathcal{D}(\mathcal{G})\triangleq\mathrm{diag}(d)$, $d=[d_1,\ldots,d_n]^{\mathrm{T}}$. A path $i_0i_1\ldots i_L$ is a finite sequence of nodes such that $i_{k-1}\sim i_k$, $k=1,\ldots,L$, and a graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. The adjacency matrix of a graph $\mathcal{G}$, $\mathcal{A}(\mathcal{G})\in\mathbb{R}^{n\times n}$, is given by $[\mathcal{A}(\mathcal{G})]_{ij}\triangleq 1$ if $(i,j)\in\mathcal{E}_\mathcal{G}$ and $[\mathcal{A}(\mathcal{G})]_{ij}\triangleq 0$ otherwise. The Laplacian matrix of a graph, $\mathcal{L}(\mathcal{G})\in\overline{\mathbb{R}}_+^{n\times n}$, which plays a central role in graph-theoretic treatments of networked multiagent systems, is given by

$$\mathcal{L}(\mathcal{G})\triangleq\mathcal{D}(\mathcal{G})-\mathcal{A}(\mathcal{G}). \tag{1}$$

In this note, we model a given networked multiagent system by a connected and undirected graph $\mathcal{G}$ (unless otherwise noted), where nodes and edges, respectively, represent agents and interagent communication links. Finally, the following lemma and remark are used in this note.

Lemma 1. [Lemma 3.3, 1]: Let $K=\mathrm{diag}(k)$, $k=[k_1,k_2,\ldots,k_n]^{\mathrm{T}}$, $k_i\in\overline{\mathbb{Z}}_+$, $i=1,\ldots,n$, and assume that at least one element of k is nonzero. Then, $\mathcal{F}(\mathcal{G})\triangleq\mathcal{L}(\mathcal{G})+K\in\mathbb{R}_+^{n\times n}$ and $\det(\mathcal{F}(\mathcal{G}))\neq 0$ for the Laplacian of a connected and undirected graph.

As a direct consequence of Lemma 1, note that because $-\mathcal{F}(\mathcal{G})$ is a symmetric and Hurwitz matrix, it follows from the converse Lyapunov theory [43] that there exists a unique $P\in\mathbb{R}_+^{n\times n}$ satisfying

$$R=\mathcal{F}(\mathcal{G})P+P\mathcal{F}(\mathcal{G}) \tag{2}$$

for a given $R\in\mathbb{R}_+^{n\times n}$.

Remark 1: Based on the notion from, for example, [44, Sec. 1.1.1.4], let $\xi(t)$ denote a solution to the dynamical system given by

$$\dot{x}(t)=f(t,x(t)),\quad x(0)=x_0. \tag{3}$$

In addition, let $t=\theta(s)$ denote the time transformation, where $\theta(s)$ is a strictly increasing and continuously differentiable function, and define $\psi(s)\triangleq\xi(t)$. Then,

$$\psi'(s)=\theta'(s)f(\theta(s),\psi(s)),\quad \psi(\theta^{-1}(0))=x_0 \tag{4}$$

where $\psi'(s)\triangleq \mathrm{d}\psi(s)/\mathrm{d}s$ and $\theta'(s)\triangleq\mathrm{d}\theta(s)/\mathrm{d}s$.

III. Problem Formulation

This section presents the proposed smooth finite-time distributed control architecture for the cooperative engagement problem.

Specifically, consider a system of n agents exchanging information among each other using their local measurements according to a connected and undirected graph $\mathcal{G}$. In addition, assume that a subset of agents can sense the position of a time-varying target; that is,

$$p(t) = \int_0^t v(\tau)\mathrm{d}\tau + p(0), \quad p(t) \in \mathbb{R}^l, \quad l \in \{1, 2, 3\} \tag{5}$$

with $v(t)$ denoting the velocity of this target.[3] In what follows, we assume for the brevity of the exposition and without loss of generality that the target has a one-dimensional position; that is, $l = 1$, since the proposed algorithm to be presented can be applied as is to multiple dimensions; that is, $l > 1$ (we refer to Section V for a numerical example). We also assume that the target's velocity is piecewise continuous and bounded.[4]

For the finite-time cooperative engagement problem, we focus on driving the positions of each agent to the position of the time-varying target in *a-priori* given, user-defined finite time T (henceforth, T is referred as the finite-time convergence constant).[5,6] Motivated from this standpoint, we propose the distributed control algorithm given by[7,8]

$$\dot{x}_i(t) = -\frac{\alpha}{T-t}\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i\,(x_i(t) - p(t))\right),$$
$$x_i(0) = x_{i0} \tag{6}$$

where $x_i(t) \in \mathbb{R}$ denotes the position of agent i, $i = 1, \ldots, n$, and $\alpha \in \mathbb{R}_+$ with (6) being defined on $t \in [0, T)$. In (6), $k_i = 1$ for a subset of agents that can sense the position of the target and $k_i = 0$ otherwise. In addition, the right-hand side of (6) denotes the local control signals of each agent; that is, $u_i(t)$, $i = 1, \ldots, n$. Mathematically speaking, we are interested in finding conditions that guarantee

$$\lim_{t\to T}(x_i(t) - p(t)) = 0, \quad i = 1, \ldots, n \tag{7}$$

with bounded control signals.

For this purpose, we first transform the proposed smooth finite-time distributed control algorithm given by (6) from $t \in [0, T)$ regular time interval to $s \in [0, \infty)$ stretched time interval in the next section, where we consider

$$\theta(s) \triangleq T\left(1 - \mathrm{e}^{-s}\right) \tag{8}$$

to utilize Remark 1. We then use tools and methods from Lyapunov stability theory for showing cooperative engagement on the new time interval with respect to s, which consequently implies finite-time cooperative engagement on the regular time interval with respect to t in the sense of (7). This is because the solution to (6) on the stretched time interval with $s \to \infty$ is equal to its solution on the regular time interval with $t \to T$.

It is also worth mentioning here that the selection of $\theta(s)$ in (8) gives $\theta'(s) = T - t$, which explicitly appears in (6). That is, although we choose a specific time transformation function $\theta(s)$ given by (8) in this note that yields to the term "$T - t$" in (6), this is for the brevity of the exposition. In general, many other time transformation functions can be chosen for the stretched time interval with $s \in [0, \infty)$ for addressing finite-time cooperative engagement problem and similar analysis steps shown in the next section can be used for each of the resulting smooth distributed control algorithms.

IV. Stability Analysis

We now show the stability analysis of the proposed smooth distributed control algorithm in Section III for the cooperative engagement problem, which include its finite-time convergence properties as well as the boundedness of local control signals of each agent. In particular, because we are interested in driving the positions of each agent to the position of the time-varying target in *a-priori* given, user-defined finite-time T, consider the state transformation given by

$$\tilde{x}_i(t) \triangleq x_i(t) - p(t) \tag{9}$$

where it follows from (6) and (9) that

$$\dot{\tilde{x}}_i(t) = -\frac{\alpha}{T-t}\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right) - v(t)$$
$$\tilde{x}(0) = \tilde{x}_{i0}. \tag{10}$$

Letting

$$\tilde{x}(t) \triangleq [\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)]^{\mathrm{T}} \in \mathbb{R}^n \tag{11}$$

and $\mathcal{F}(\mathcal{G}) \triangleq \mathcal{L}(\mathcal{G}) + K$, where $K = \mathrm{diag}(k)$, $k = [k_1, k_2, \ldots, k_n]^{\mathrm{T}}$, (10) can be rewritten in a compact form

$$\dot{\tilde{x}}(t) = -\frac{\alpha}{T-t}\mathcal{F}(\mathcal{G})\tilde{x}(t) - \mathbf{1}_n v(t), \quad \tilde{x}(0) = \tilde{x}_0. \tag{12}$$

From Lemma 1, note that $-\mathcal{F}(\mathcal{G})$ is a symmetric and Hurwitz matrix, and hence there exists a unique $P \in \mathbb{R}_+^{n\times n}$ satisfying (2) for a given $R \in \mathbb{R}_+^{n\times n}$.

Next, let $\xi(t)$ denote a solution to the dynamical system given by (12). In addition, let $t = \theta(s)$ denote the time transformation, where $\theta(s)$ is given by (8) with $s \in [0, \infty)$ being the stretched time interval, and define $\psi(s) \triangleq \xi(t)$. It then follows from Remark 1 that

$$\begin{aligned}\psi'(s) &= \theta'(s)\left(-\frac{\alpha}{T - T(1-\mathrm{e}^{-s})}\mathcal{F}(\mathcal{G})\psi(s) + w(s)\right)\\ &= T\mathrm{e}^{-s}\left(-\frac{\alpha}{T\mathrm{e}^{-s}}\mathcal{F}(\mathcal{G})\psi(s) + w(s)\right)\\ &= -\alpha\mathcal{F}(\mathcal{G})\psi(s) + T\mathrm{e}^{-s}w(s),\ \psi(\theta^{-1}(0)) = \tilde{x}_0\end{aligned} \tag{13}$$

where $w(s) \triangleq -\mathbf{1}_n v(\theta(s))$ and $\theta^{-1}(0) = 0$. Furthermore, letting $\mu(s) \triangleq T\mathrm{e}^{-s}$, $\mu(s) \in \mathbb{R}$, it follows from (13) that

$$\psi'(s) = -\alpha\mathcal{F}(\mathcal{G})\psi(s) + \mu(s)w(s) \tag{14}$$
$$\mu'(s) = -\mu(s) \tag{15}$$

where $\mu(0) = T$. Note that (14) and (15) do not explicitly depend on s on the stretched time interval—unlike (12) that explicitly depends on t on the regular time interval. To this end, we use the standard Lyapunov stability theory in the next theorem in order to show cooperative engagement on the new time interval with respect to s, which consequently implies finite-time cooperative engagement on the regular time interval with respect to t, as discussed earlier.

Theorem 1: Consider a networked multiagent system, where agents exchange information using their local measurements according to a connected and undirected graph $\mathcal{G}$. In addition, consider that each agent

[3] Theoretically, the dimension of $p(t)$ can be any finite number.

[4] Unlike [23]–[25], we do not assume the knowledge of the upper bound on the target's velocity, and hence here it is treated as unknown.

[5] Unlike [34]–[41], we address finite-time convergence both on a dynamic equilibrium and with bounded controls, where the difference between the positions of each agent and time-varying target defines this dynamic equilibrium.

[6] Motivated by the nature of the cooperative engagement problem, we consider smooth distributed control design over the user-defined time interval $[0, T)$ throughout this note—owing to the fact that the task under consideration completes once the agents and the target coincide as $t \to T$.

[7] In (6), it is assumed that α and T are available to the agents.

[8] The term "$\sum_{i\sim j}(x_i(t) - x_j(t))$" used in (6) is generally referred as the consensus term, where it is common in many finite-time as well as infinite-time distributed control algorithms (see, for example, books [2], [3]).

utilizes the distributed control algorithm given by (6), where there exists at least one agent that can sense the position of a time-varying target with bounded but unknown velocity. Then, (7) holds for all initial conditions of agent positions.

Proof: We consider the Lyapunov function candidate

$$V(\psi, \mu) = \psi^{\mathrm{T}} P \psi + c\mu^2 \tag{16}$$

where $c \in \mathbb{R}_+$ and $P \in \mathbb{R}_+^{n\times n}$ is the solution to (2) for a given $R \in \mathbb{R}_+^{n\times n}$. Note that $V(0,0) = 0$ and $V(\psi,\mu) > 0$ for $(\psi,\mu) \neq (0,0)$. Differentiating (16) with respect to $s \in [0,\infty)$ along trajectories of (14) and (15) for any $\psi(0)$ and $\mu(0)$ yields

$$\begin{aligned} V'(\psi(s),\mu(s)) &= 2\psi^{\mathrm{T}}(s) P\left(-\alpha\mathcal{F}(\mathcal{G})\psi(s) + \mu(s)w(s)\right) \\ &\quad + 2c\mu(s)\left(-\mu(s)\right) \\ &= -\alpha\psi^{\mathrm{T}}(s)R\psi(s) + 2\mu(s)\psi^{\mathrm{T}}(s)Pw(s) \\ &\quad - 2c\mu^2(s) \end{aligned} \tag{17}$$

where $V'(\psi(s),\mu(s)) \triangleq \mathrm{d}V(\psi(s),\mu(s))/\mathrm{d}s$. Applying Young's inequality [45] to the term "$2\mu(s)\psi^{\mathrm{T}}(s)Pw(s)$" in (17) results in

$$\begin{aligned} 2\mu(s)\psi^{\mathrm{T}}(s)Pw(s) &\leq 2\lambda_{\max}(P)|\mu(s)|\|\psi(s)\|_2 w^* \\ &\leq \frac{1}{d}\|\psi(s)\|_2^2 + d|\mu(s)|^2 \cdot \lambda_{\max}(P)^2 w^{*^2} \end{aligned} \tag{18}$$

where $d \triangleq (\lambda_{\max}(P)^2 w^{*2})^{-1} c \in \mathbb{R}_+$. Here, note that $\|w(s)\|_2 \leq w^*$ used in (18) follows from the boundedness of the velocity of the target. ■

Next, it follows from (17) and (18) that

$$\begin{aligned} V'(\psi(s),\mu(s)) &\leq -\|\psi(s)\|_2^2\left(\alpha\lambda_{\min}(R) - d^{-1}\right) \\ &\quad - |\mu(s)|^2\left(2c - d\lambda_{\max}(P)^2 w^{*^2}\right) \\ &= -\|\psi(s)\|_2^2\left(\alpha\lambda_{\min}(R) - c^{-1}\cdot\lambda_{\max}(P)^2 w^{*^2}\right) \\ &\quad - c|\mu(s)|^2. \end{aligned} \tag{19}$$

Now, setting $c \triangleq 2(\alpha\lambda_{\min}(R))^{-1}\lambda_{\max}(P)^2 w^{*2} \in \mathbb{R}_+$ yields

$$V'(\psi(s),\mu(s)) \leq -\frac{\alpha\lambda_{\min}(R)}{2}\|\psi(s)\|_2^2 - c|\mu(s)|^2 \tag{20}$$

and hence, the dynamics given by (14) and (15) are the Lyapunov stable for all $\psi(0)$ and $\mu(0)$ on stretched time interval $s \in [0,\infty)$ and $\lim_{s\to\infty}(\psi(s),\mu(s)) = (0,0)$ holds [43, Th. 4.6]. Finally, since (14) and (15) together give (13) with $\mu(0) = T$, $\psi(s) = \xi(t)$ by definition with $\xi(t)$ being the solution to (12), and $t \to T$ as $s \to \infty$ by (8), the result given by (7) is now immediate for all initial conditions of agent positions. ■

Note that Theorem 1 shows not only the convergence of the positions of each agent to the position of the time-varying target in *a-priori* given, user-defined finite-time T in the sense of (7) without dependence on the initial conditions of these agents, but also the boundedness of the positions of agents on the regular time interval $t \in [0,T)$ through the Lyapunov stability on the stretched time interval $s \in [0,\infty)$. As discussed earlier, letting the right-hand side of (6) be the local control signals of each agent; that is, $u_i(t)$, $i = 1,\ldots,n$, the next theorem shows the condition on the boundedness of $u_i(t)$, $i = 1,\ldots,n$.

Theorem 2: Consider a networked multiagent system, where agents exchange information using their local measurements according to a connected and undirected graph $\mathcal{G}$. In addition, consider that each agent utilizes the distributed control algorithm given by (6), where there exists at least one agent that can sense the position of a time-varying target with bounded but unknown velocity, and assume

$$\mathcal{S} \triangleq \alpha\mathcal{F}(\mathcal{G}) - \mathrm{I}_n \in \mathbb{R}_+^{n\times n}. \tag{21}$$

Then, the local control signals of each agent are bounded.

Proof: Letting $u(t) \triangleq [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$, it follows from (6) and (9) that

$$u(t) = -\frac{\alpha}{T-t}\mathcal{F}(\mathcal{G})\tilde{x}(t) \tag{22}$$

where differentiating (22) with respect to $t \in [0,T)$ yields

$$\begin{aligned} \dot{u}(t) &= -\frac{\alpha}{T-t}\mathcal{F}(\mathcal{G})\dot{\tilde{x}}(t) - \frac{\alpha}{(T-t)^2}\mathcal{F}(\mathcal{G})\tilde{x}(t) \\ &= -\frac{1}{T-t}\mathcal{S}u(t) - \frac{\alpha}{T-t}\mathcal{F}(\mathcal{G})(-\mathbf{1}_n v(t)). \end{aligned} \tag{23}$$

■

Next, let $\phi(t)$ denote a solution to (23). Furthermore, let $t = \theta(s)$ denote the time transformation, where $\theta(s)$ is given by (8) with $s \in [0,\infty)$ being the stretched time interval, and define $\rho(s) \triangleq \phi(t)$. Then, using identical steps shown earlier in this section, it follows from Remark 1 that

$$\rho'(s) = -\mathcal{S}\rho(s) - \alpha\mathcal{F}(\mathcal{G})w(s) \tag{24}$$

where $\rho'(s) \triangleq \mathrm{d}\rho(s)/\mathrm{d}s$. Note that the term "$\alpha\mathcal{F}(\mathcal{G})w(s)$" in (24) is bounded by the boundedness of the velocity of the target. Note also that $-\mathcal{S}$ is Hurwitz from (21). Therefore, it follows from input-to-state stability [43, Sec. 4.5] that $\rho(s)$ is a bounded solution to the dynamical system given by (24) on the stretched time interval $s \in [0,\infty)$. Finally, since $\rho(s) = \phi(t)$ by definition, where $\phi(t)$ is the solution to (23), and $t \to T$ as $s \to \infty$ by (8), the result is now immediate. ■

The boundedness of the local control signals of each agent is shown in Theorem 2 subject to (21); that is, $\|u(t)\|_2 \leq u^*$ with $u(t)$ given by $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]$. From a practical standpoint, it should be mentioned based on (24) that u^* increases as α in (6) increases as well as the magnitude of the velocity of the time-varying target increases.[9] Finally, the next corollary shows how the results of Theorems 1 and 2 hold in the case of connected and directed graphs.

Corollary 1: Consider a networked multiagent system, where agents exchange information using their local measurements according to a directed graph $\mathcal{G}$ with this graph having a spanning tree[10] from the agents that can sense the time-varying target to the rest. In addition, consider that each agent utilizes the distributed control algorithm given by (6), where there exists at least one agent that can sense the position of a time-varying target with bounded but unknown velocity. Then, (7) holds for all initial conditions of agent positions. Finally, if $\mathcal{M}$ is Hurwitz with $\mathcal{M} \triangleq -\alpha\mathcal{F}(\mathcal{G}) + \mathrm{I}_n \in \mathbb{R}^{n\times n}$, then the local control signals of each agent is bounded.

Proof: For a connected and directed graph $\mathcal{G}$ subject to the spanning tree assumption, we first note that $-\mathcal{F}(\mathcal{G})$ is Hurwitz (see [1, Lemma 3.3]). In this case, since $-\mathcal{F}(\mathcal{G})$ is not necessarily symmetric, we replace the Lyapunov equation given by (2) with $R = \mathcal{F}^{\mathrm{T}}(\mathcal{G})P + P\mathcal{F}(\mathcal{G})$. Then, based on this Lyapunov equation, (7) readily follows from the proof of Theorem 1. Finally, following similar steps used in the proof of Theorem 2, one can write $\rho'(s) = \mathcal{M}\rho(s) - \alpha\mathcal{F}(\mathcal{G})w(s)$ in this case. The boundedness of the local control signals is now immediate since $w(s)$ is bounded and $\mathcal{M}$ is Hurwitz. ■

[9]While the calculation of u^* is not presented, this can be readily done using, e.g., perturbed systems theory [46] or $\mathcal{L}$ systems theory [47].

[10]A directed graph has a spanning tree if there exists a root node such that it has directed paths to all other nodes in the graph (see, e.g., books [1]–[3]).

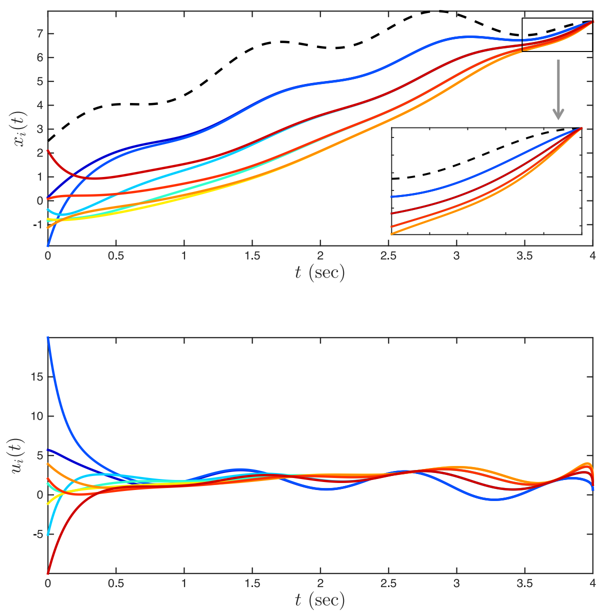

Fig. 1. Response of a networked multiagent system subject to random initial conditions with the smooth finite-time distributed control algorithm given by (6) with $T = 4$ seconds and $\alpha = 10$ (one-dimensional case), where the solid lines show the positions (top) and the control signals (bottom) of each agent and the dashed line shows the position of the target.

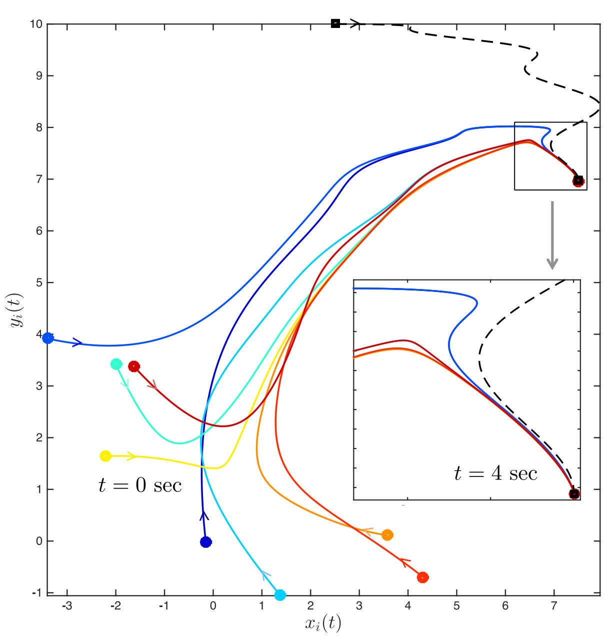

Fig. 2. Response of a networked multiagent system subject to random initial conditions with the smooth finite-time distributed control algorithm given by (6) with $T = 4$ seconds and $\alpha = 10$ (two-dimensional case), where the solid lines show the planar positions of each agent and the dashed line shows the planar position of the target.

V. Illustrative Numerical Example

An illustrative numerical example is presented in this section to demonstrate the efficacy of the results in Sections III and IV. Specifically, consider a networked multiagent system consisting of eight agents; that is, $n = 8$, subject to a connected and undirected circle graph, where the first two agents on this graph can sense the position of a time-varying target; that is, $k_i = 1$ for $i = \{1, 2\}$ and $k_i = 0$ for $i = \{3, \ldots, 8\}$ in (6). In addition, we choose the finite-time convergence constant as $T = 4$ seconds and set $\alpha = 10$ in (6), which satisfies the assumption in Theorem 2 given by (21). In what follows, random initial conditions are considered for the agents.

Fig. 1 shows the response of this networked multiagent system in a one-dimensional setting with the smooth finite-time distributed control algorithm given by (6), where the solid lines show the positions (top) and the control signals (bottom) of each agent and the dashed line shows the position of the target given by $p(t) = 2.5 + 5\sin(0.5t) + 0.5\sin(5t)$. As expected from Theorem 1, the positions of each agent converges to the position of the time-varying target at $T = 4$ seconds in the sense of (7). In addition, as expected from Theorem 2, the control signals of each agent remains bounded. As noted earlier in this note, the proposed distributed control algorithm in (6) can be applied as is to multiple dimensions. To this end, Fig. 2 shows the responses of the same networked multiagent system in a two-dimensional setting, where the solid lines show the planar positions of each agent and the dashed line shows the planar position of the target given by $p(t) = [2.5 + 5\sin(0.5t) + 0.5\sin(5t), 10\cos(0.2t)]^{\mathrm{T}}$. Specifically, in this case, the proposed distributed control algorithm given by (6) is implemented for each dimension simultaneously. Once again, as expected, the planar positions of each agent converges to the planar position of the time-varying target at $T = 4$ seconds.

VI. Conclusion

For contributing to the previous studies in networked multiagent systems, we proposed and analyzed a smooth finite-time distributed control architecture using a time transformation method and Lyapunov stability theory. Specifically, under the assumption that a subset of agents can sense the position of a time-varying target, it was shown that the proposed architecture guarantees finite-time cooperative engagement in that the difference between the positions of each agent and this target vanishes in *a-priori* given, user-defined finite time without dependence on the initial conditions of agents, and in the presence of unknown but bounded velocity of this target. While this note focuses on a particular problem in the context of multiagent systems, the proposed time transformation method and the analysis procedure can be used for many other problems, where *a-priori* given, user-defined finite-time convergence is necessary with smooth control laws.

To this end, based on the proposed time transformation method and the analysis procedure, future research will include, first, generalizations of our framework to agents having high-order (linear and nonlinear) dynamics, and second, graph topologies that vary with respect to time, where the results reported in, for example, [48] can be useful on the latter generalization under a relaxed condition. In addition, following the results reported in [49], we will consider, third, the communication constraints among agents, such as transmission nonlinearity and time-varying time-delays. Another future research direction will be, fourth, the consideration of measurement noise in the real-world execution of the proposed framework. In particular, since it has high gain as time approaches to T, one can consider low-bandwidth discretized implementations in the presence of (excessive) measurement noise, where this is motivated by the results in, for example, [50]. To this end, it will be interesting to reveal the tradeoff between the sampling time selection for discretized implementations and the system performance. Finally, we will also consider extensions including, fifth, the preservation of the graph connectivity, and finally, the avoidance of possible interagent collisions during the real-world execution of the proposed multiagent systems framework over $t \in [0, T)$.

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